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Grammar-based Rhombic Polyhedral Multi-Directional Joints and Corresponding Lattices

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Abstract

This paper presents a new type of joint derived from traditional Chinese and Japanese wooden puzzles. A rule-based computational grammar for generating the joints is developed based on the relationship between the three-dimensional shape and its two-dimensional "net". The rhombic polyhedra family, cube, rhombic dodecahedron, and rhombic triacontahedron are fully explored in the paper as they are the central part of the joints and is enclosed by all the elements around. This paper also presents lattices derived from these joints, and provides structure analysis to compare the structural performance of these lattices. The non-orthogonal lattices provide new way of looking at lattice structures and also provide new challenges and opportunities for designing assembly techniques by hand, robotics, and drones.

Keywords: polyhedra, lattice, structural joint, drone-based assembly

1. Introduction

Conventional building structures rely on large-scale construction material systems such as reinforced concrete and structural steel framing to gain sufficient strength and stiffness. They need either on-site casting or prefabrication, the latter of which also requires large equipment to install. Furthermore, these construction techniques consume a large amount of energy and are often non-reversible construction methods. To respond to the need for a sustainability building industry and energy-sensitive construction environment, existing means of construction and assembly systems need to improved, and new ways need to be invented.

The increasing developments of robotics-based fabrication and construction is contributing significantly to this field. The change of tools often affects the change of related methodologies to suit the way these tools are used. Robotic fabrication systems such as robotic arms, drones, etc. are able to achieve works with higher degree complexity in a faster, more efficient way, but at the same time are limited to the payload they can support, which requires assembly-like structural systems. Each member of the system needs to small and light enough to be handled and must be connected with strong enough joints to form a larger-scale holistic structure. Meanwhile, realization of different algorithms through computation not only enables robots and drones to achieve their work but also provides more freedom and opportunities to design structures with higher degree of complexity and that are difficult to assemble by conventional human labour. These algorithms also populate the number of design methods and allow us to design in unprecedented ways. For instance, computational shape grammars are available to design and generate building units for unconventional structures parametrically.

From the perspective of structural efficiency, where the goal is to support more load with less materials, one well-established approach is the lattice-based structure, including trusses, tensegrity structures, deployable and foldable structures, etc. However, many structures of this type are orthogonal (some with diagonal members as triangulation). From a historical perspective, these types have simpler logic that are easier to understand and can be designed for variation easily and quickly. Since all members are either vertical or horizontal, it is also easier to model, analyze and simulate dynamically in finite element software. The design of the joints can also be simplified to fewer types as all members came from the same direction.

Meanwhile, non-orthogonal lattice structures, which have received attention and development in a broad set of sub-fields (chemistry, electronic engineering, etc.), have not been developed much as structural systems for reasons of constructability. The difficulties may lie in the design of joints, assembly process, the tolerance, etc. But new tools and methods we are engaging now provide an opportunity to return to this concept and learn from old techniques, and discover new possible high-performing and architecturally interesting applications.

Figure 1: Kongming Lock [image from http://de.wikipedia.org/wiki/Bild:SechsTeileHolzknoten.jpg]

This paper starts from one of the most ancient wooden puzzle in Asia, the "Kongming Lock" (Figure 1), a derivative of building joints for Chinese traditional wooden architecture, and mathematically and computationally presents the logic and rules for the generation of the puzzle. The paper also discovers the grammars that generate the only other two more complex derivatives of the same type. By understanding the relationship of different members, new non-orthogonal lattices will be generated and presented at the end of this paper.

2. Literature review

2.1 Polyhedron and lattice

The application of polyhedra in structural design has broadly explored in different scales, ranging from the creation of the roundest soccer ball (Huybers [1]) to building scale polyhedral domes (Wester [2]). Lattice structures formed by some of these polyhedra can also achieve free-form structure to suit for the architectural need (Davis [3]). Variations of polyhedral structures such as geodesic domes have also been built and patented (Buckminster [4]), the joints of which connect five and six members connecting from a morphed surface plane (Ohme [5]).

However, most of these applications of polyhedra focuses on using the polyhedra as domes, surface structures that enclose certain spaces, or on producing space trusses composed of repetitive polyhedral geometries. Few has focused on using polyhedra as joints to generate complex lattice structures whose members are placed towards multiple directions other than orthogonal ones. Some special polyhedral joints were designed to provide multi-directional connection, such as the triakonta system (Elliott [6]), and the KK system (Chilton [7]), as well as related patents (Jeannin [8]). But most of these applications either use the polyhedron as a multi-directional provider or invent special cases to suit the need. Few has been focused on the lattice structure that different polyhedral joints can provide.

2.2 Traditional Asian joints and wooden puzzles

The puzzle of "burr" refer to a series of puzzle families composed of different members interlocked with each other by carefully cut notches. The most common type of the puzzle is the six-piece burr, also called “Puzzle Knot”, “Chinese Cross” or “Kongming Lock”. It is the most well-known and presumably the oldest of the burr puzzles. It is actually a family of puzzles with the same finished geometry and basic shape of the pieces, with different notches. This family of puzzles was first solved mathematically and completely by mathematician Bill Cutler in 1994 with the help of computational analysis, as in Cutler [9]. The notches on the pieces for each of these puzzles are often different from each other, resulting in different assembly processes.

Other burr families, such as Altekruse, Chuck, Pagoda, are also developed throughout the time (Coffin [10]). These puzzles in general can be divided into two categories: puzzles whose members share the same basic shape (Figure 1) and puzzles whose members share different basic shapes (Figure 2).

Figure 2: A puzzle with different shaped members [image from https://commons.wikimedia.org/wiki/File:Pagoda_Burr_Puzzle.jpg]

Most of the burr puzzles are made with square notches. There is a small category of puzzles are made with diagonal notches, often called “stars” (Slocum [11]). However, careful study of the bounding boxes of the members of these puzzles shows they are actually the same type of the puzzles above.

Most of the puzzles were made through physical experiments by cutting/gluing (Coffin [12]) blocks. For complex puzzles, this process is exhausting and even impossible as it is hard to keep the tolerance down after multiple cuts. Moreover, there are few general tools to understanding the geometric and assembly logic behind these puzzles. Since most of these puzzles are symmetric in space, it is reasonable to infer certain rule based systems will help in computing these geometries through iterative processes.

2.3 Grammatical computation

As a well-known type of rule-based design method in design and computation field, shape grammars is a commonly used method to generate multiple design possibilities by defining a set of allowable shape transformations. Once formulated, the rules can automatically generate new designs diversely. Stiny has provided many two-dimensional examples in his book "Shape: Talking about Seeing and Doing" (Stiny [13]). Esher’s published tile designs are also examples for 2D tessellation and are examined in a computational perspective on analytical methods for design (Ozgan *et al*. [14]). 3D grammars and algorithms have also been developed to analysis polyhedral object pattern (Wang [15]), or to extend the frames in conceptual design representation (Albert *et al.* [16]). More advanced implementations of shape grammars for convex polyhedra adapt grammar-based tools to wider range of geometries (Thaller *et al.* [17]).

2.4 Research question

The topological design of almost all lattice structure is actually a 3D tessellation problem about how to use basic polyhedral units to occupy the space. Research and application in reality are mostly focused on cubic units which results in orthogonal-looking structures that are both easier to understand and construct. However, thanks to the increasing use of compuation and digital fabrication methods such as robotic fabrication and drone-based construction, lattices with higher complexity and constructability are now easier to understand, fabricate, and build. These structures may even benefit the process and logic of these fabrication and construction methods if designed carefully to suit the machines. This paper will discuss (1) Grammatical algorithms for traversing faces of three rhombic polyhedrons: cube (regular hexahedron), rhombic dodecahedron, rhombic triacontahedron; (2) Space lattice structures derived from the joints of these three rhombic polyhedrons; (3) structural assessments of these lattices.

3. Shape Grammar for Traversing Polyhedra

3.1 6-piece geometric puzzle and shape grammar

The puzzle in Figure 1 can be regarded as a cluster of six sticks intersecting each other. An interlocking state can be achieved by removing parts from each sticks. Since the number of ways that sticks can be arranged symmetrically and spatially is very limited, it is useful to examine the question with unnotched members.

By examining this the puzzle, a center cube (Figure 3) can be found as the intersection part shared by all the six members. The six members each corresponds to one of the six faces, and a grammar that demonstrates the relationship (the rule) between every two members can be found. By computing the same grammar repetitively, we can generate the overall geometry of the puzzle after 6 iterations (Figure 4).

Figure 3: Central cube of the puzzle

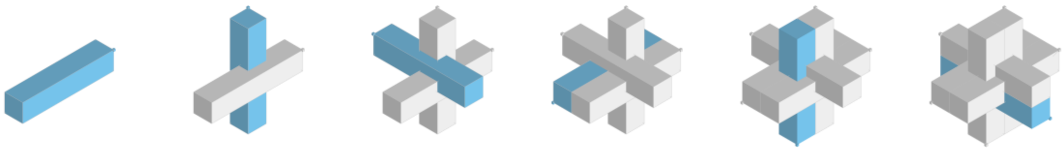


Figure 4: Computation process

During the computation process, the six corresponding faces of the center cube are also traversed with no repetition. No matter how the geometric shape of the six members changes, the relationship between each member and its corresponding face as well as the grammar will maintain the same. Thus, the grammar for computing the shapes is equivalent to a grammar that traverses all the faces of a cube without overlapping.

3.2 “Net” and 2D computation

By unfolding the cube into a planar diagram, which is called a “planar net” (Buekenhout & Parker [18]), the grammar can be transformed from 3D to 2D. For a cube, there are in total 11 distinct nets (Figure 5) that exist (Weisstein [19]) and more than one grammar can be defined for each net based on the number of rules in each grammar. The single grammar mentioned above belongs to the 6th net, which visually has a repetitive relationship between every two connected squares.

Following the shape grammar defined in Figure 6, we can easily compute the result as shown in Figure 7. Notice here that though only 6 steps are shown here as we are computing the 6th net of the cube, this iteration can continue infinitely if only considering the 2D layout.

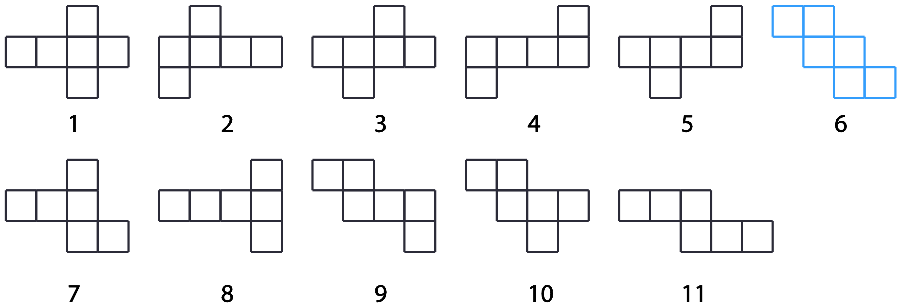
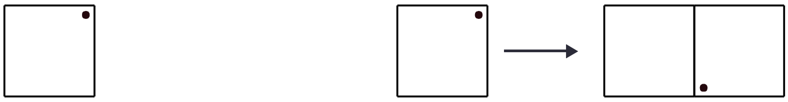


Figure 5: Eleven distinct nets for the cube



a. initial shape   b. shape rule (with label)

Figure 6: One-rule grammar for computing net traversal

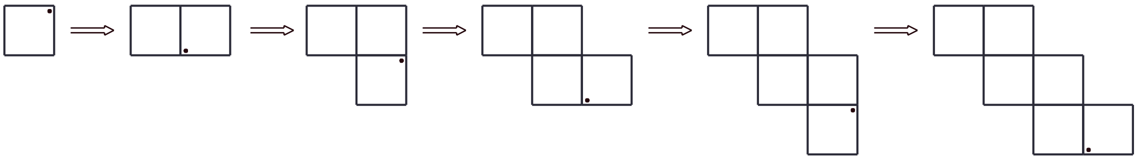
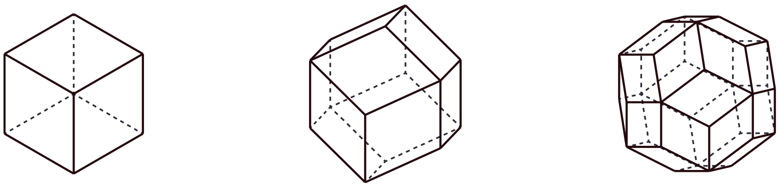


Figure 7: Grammar computation

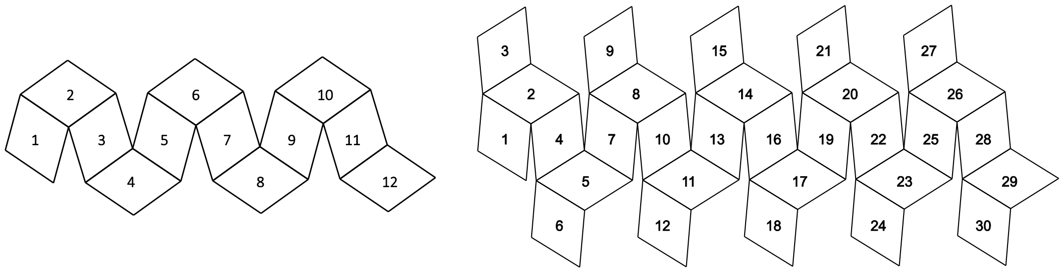
The number of distinct nets increases extremely fast depending on the number of faces of the polyhedron. For instance, the dodecahedron and icosahedron each have 43,380 distinct nets (Buekenhout & Parker [18]).

In the book *Geometric Puzzle Design*, Coffin [10] notes that faces of the enclosed center must be rhombic (square is a special case of rhombic), and there are only three isometrically symmetrical solids with such faces: the cube, the rhombic dodecahedron, and the rhombic triacontahedron (Figure 8). Due to the exhaustive number of nets for these shapes, this paper will not examine all the distinct nets of the rhombic dodecahedron and the rhombic triacontahedron, but will only show one of the nets that has a grammar with a minimal number of rules (Figure 9).



a. Cube   b. Rhombic dodecahedron   c. Rhombic triacontahedron

Figure 8: The only three polyhedra with rhombic faces



a. Rhombic dodecahedron b. Rhombic triacontahedron

Figure 9: Nets and computation sequence for rhombic dodecahedron and rhombic triacontahedron

3.3 symmetrical geometry computed from 3D shape grammar

If the grammar is computed in 3D space, the computation will traverse all the faces on the three polyhedra respectively. The corresponding linear members of each face of the polyhedron will compose a complex spatial geometry (Figure 10) in which no member is intersected with any other member. The space enclosed by all the members is the corresponding polyhedron.

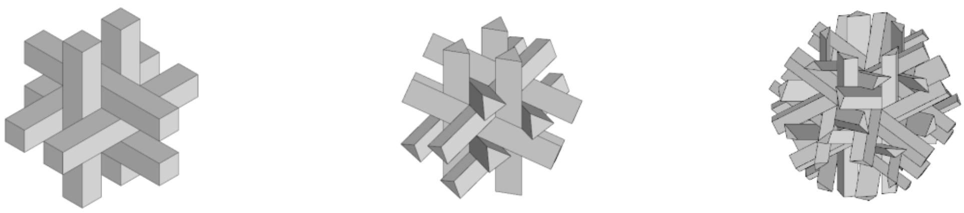


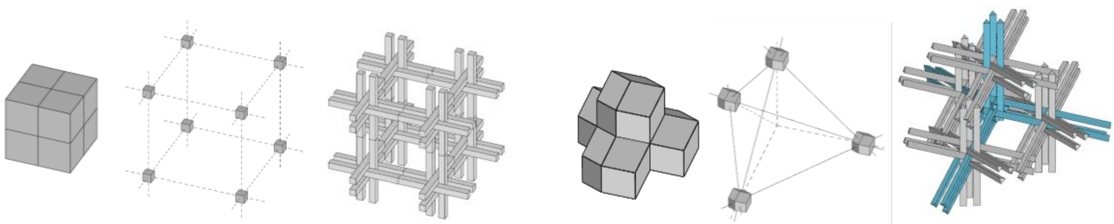
Figure 10: Geometries composed by linear elements corresponding to the faces of the central polyhedra

4 Lattice with polyhedral joints

Most of the existing research or projects focus on developing polyhedra as the primitive shape of dome structures or extending the edges of polyhedra to form space frames. In this paper, we will discuss the possibility of forming space lattices by using the three polyhedra mentioned above as joints, and extending the linear members corresponding to the faces of these polyhedra. This will produce spatial lattices, some of which will have non-orthogonal connected space units.

4.1 Lattice of cubic joints

As the faces of the cube are oriented orthogonally, the lattice produced by the joints based on the first geometry in Figure 8a will still be an orthogonal style. The difference is that there are two parallel linear elements coming from each joint in each direction, and in total 6 members oriented in 3 directions – as the result of 3 pairs of parallel faces in the cube. By orienting multiple cubes into a 3D matrix, we can create lattices from these cubic joints by adding linear members along their faces (Figure 11.a). This type of lattice can be extended infinitely in space and defines different geometric topologies.



a. Cubic b. Rhombic dodecahedral

Figure 11: Polyhedral tessellation, matrix, and corresponding lattice

4.2 Lattice of rhombic dodecahedral joints

Similar to the cubic case, we implement the same method to the rhombic dodecahedron joints Figure 8b). From Figure 10, it is clear that there are three parallel linear elements coming from each joint in each direction and in total 12 members oriented in 4 directions. The 3D matrix for orienting the polyhedral joints is also derived from the tessellation of the basic shape: the rhombic dodecahedron.

However, unlike the case in Section 4.1, linear members coming from neighboring do not share the same axis. Thus will not coincide collinearly as their lengths are extended; instead they will create new joint locations in space. The polyhedron enclosed in these new joints are also rhombic dodecahedron (Figure 11b). If extending continuously in space, we can also find the simplified lattice for this polyhedral joints. The lattice unit is shown in Figure 12. It is composed of the wireframe of rhombic dodecahedron and the four axes for the 2^{nd} symmetry shown in Figure 13.

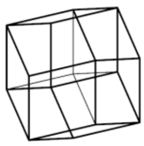


Figure 12: The unit of rhombic-dodecahedron-joint lattice

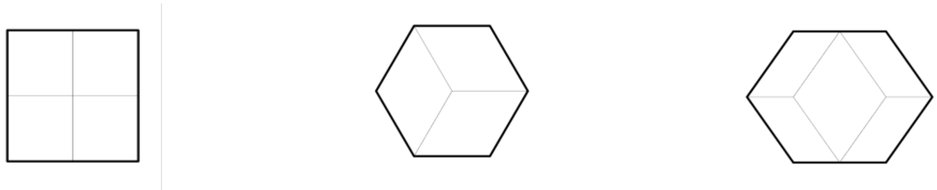


Figure 13: Three types of symmetry of rhombic dodecahedron

4.3 Rhombic Triacontahedral Joints

For rhombic triacontahedron, the circumstance is different as it is not 3D tessellation shape – unlike cube and rhombic dodecahedron, rhombic triacontahedron cannot fill three dimensional space by orienting itself.

It might be possible to combine rhombic triacontahedron and its stellation, rhombic hexecontahedron, or other polyhedra to become 3D tessellation polyhedras, but this topic is beyond the scope of this paper and may be considered in future work.

5. Structural assessment of lattices

5.1 Lattice Design

To compare the structural performance of the different lattices composed of joints proposed above, we defined a testing geometry, with dimension of 10m x 10m x 15m, filled with these lattices in different densities (Figure 14). The density is controlled by the number of subdivision of the 3D space, and is calculated as the ratio of the total volume of the targeted lattice to the total volume of the bounding box. Notice that the section area of the linear elements in each lattice may vary, due to the fixed total volume under each density. The assessment platform is created using the software platforms Rhinoceros v5.0, Grasshopper3D, and Karamba v1.1.

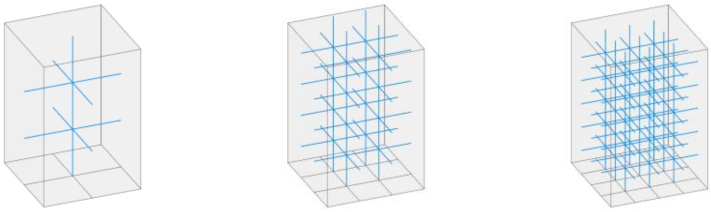


Figure 14: Testing volume filled with lattice of different density (base mesh 2x2, 3x3, 4,4)

By examining carefully about the lattices in Section 4, we find there are two parallel linear elements and three parallel linear elements in each direction coming from each joint. These pairs are not connected besides the two ends. It is reasonable to simplify these pairs into single linear beam element and modify structural properties through Karamba, i.e. moment of inertia, loading type, etc. to simulate the scenario. The simplified lattice for cube is a normal orthogonal space truss (Figure 14). The simplified lattice of rhombic dodecahedron joints has 3 topological forms due to different stacking methods, since rhombic dodecahedron has 3 symmetric axes [Figure 14]. However, only two of them can have a vertical staking topology. The third will result in an overall inclination thus will not be discussed here due to structural stability reasons (Figure 15).

a. Version 1 b. Version 2

Figure 15: Rhombic-dodecahedron-joint lattices with different stacking methods

5.2 FEA Analysis

We defined two load cases for the testing geometry: 1. vertical loads to simulate the behaviour of a column; 2. shear loads to simulate structural behaviour of a cantilever (Figure 16).

a. 100 kN surface load b. 50 kN surface load

Figure 16: Two load cases for the testing geometry.

We vary the base mesh from 4x4 to 16x16 (only even division) similar to the one in Figure 14 and fill the testing geometry with three types of lattice: the orthogonal cubic lattice (F6), and two versions of the rhombic-dodecahedron-joint lattice (F12_v1, F12_v2). By adding the two load cases on these lattices respectively, we can visualize the results in Figure 17:

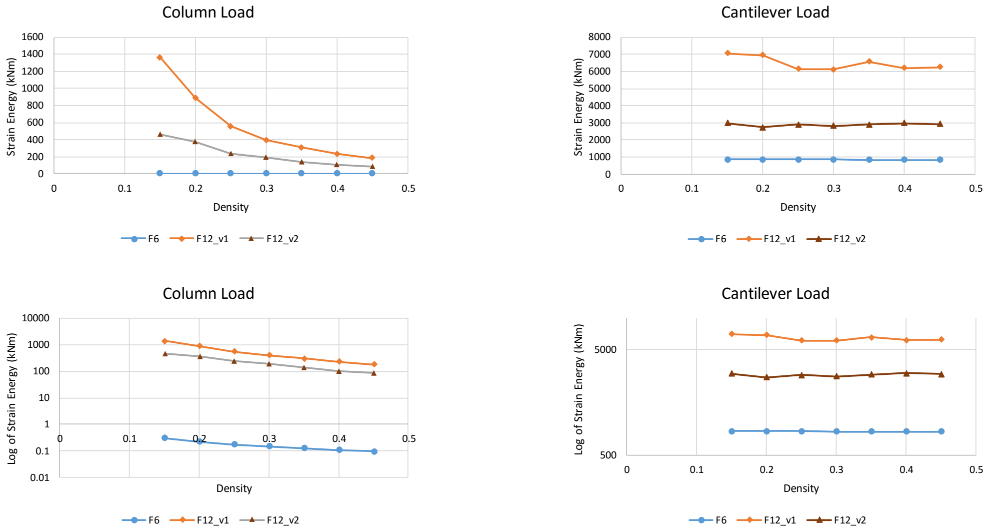


Figure 17: Results of two load cases

For both cases, the F6 lattice performs better than the two F12 lattices. Under column loading, the strain energy of all three lattices will converge as the density increases. While under cantilever load, the strain energy of each lattice will fluctuate around its certain value and there is no sign of convergence. The result shows that though designed as new type of lattices, the F12 lattices are not structurally efficient, especially when building large scale structures where the limits of the materials are more likely to reach. However, the complexity and aesthetic property of this type of lattice provides more architectural possibilities in small scale structures (Figure 18), and opportunities and challenges for the design of the joints.

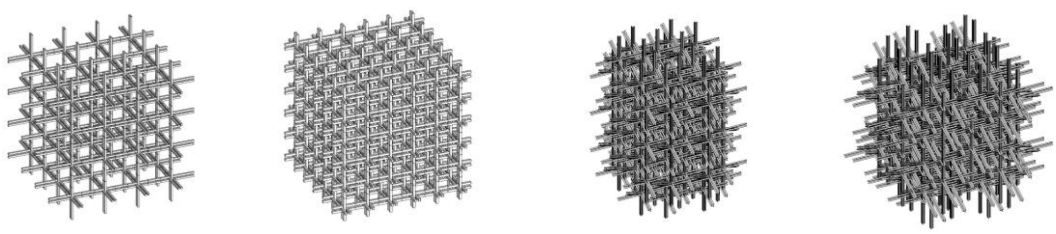


Figure 18: Lattice variations

6. Conclusion

By looking back to the historical wooden puzzle and the relationship between these puzzles and their corresponding polyhedra, this paper provides a different perspective in bridging the old wisdom with today's digital tools. It offers a computational grammar-based generation logic for a certain type of symmetric polyhedral complex geometry by bridging the 2D "net" with the 3D geometry. Unlike much polyhedral research that focuses on using the polyhedral geometry as domes, this paper provides several new kind of lattices derived from the polyhedral joints using a bottom-up logic, and assesses the structural performance of these lattices.

As a new perspective of looking at symmetric polyhedral, future work may include: the possible lattice in the rhombic triacontahedron category; the variation and application of rhombic dodecahedral lattices; the improvement of structural performance of these lattices; the design of joints/notches for connecting these complex lattices in reality.

References

[1] Huybers, I. P. Soccer ball geometry, a matter of morphology. *International Journal of Space Structures*, 22(3), 2007, 151-160.

[2] Wester, T. The structural morphology of basic polyhedra. *Beyond the Cube: The Architecture of Space Frames and Polyhedra*, Chapter 11, John Wiley & Sons, 1997, 301–342.

[3] Davis, L.Philip Johnson's Crystal Cathedral and the Rhetoric of Its Free-Form Polyhedral Structure. *Beyond the Cube: The Architecture of Space Frames and Polyhedra*, Chapter 5, John Wiley & Sons, 1997, 161–187.

[4] Buckminster, F. R. *U.S. Patent No. 2,682,235*. U.S. Patent and Trademark Office, 1954.

[5] Ohme, D. A. *U.S. Patent No. 4,370,073*. U.S. Patent and Trademark Office, 1983.

[6] Elliott, J., The Triakonta Structural System: From Toy to Tower, *Proceedings of the International Convention of Society of Wood Science and Technology and United Nations Economic Commission for Europe – Timber Committee*, Geneva, 2010.

[7] Chilton, J., Space Grid Geometry: Thinking in Three Dimensions. *Space grid structures*, Chapter 2, Taylor & Francis, 2007.

[8] Jeannin, J. L. *U.S. Patent No. 4,247,218*. U.S. Patent and Trademark Office, 1981.

[9] Cutler W. H. A computer analysis of all 6-piece burrs. *Self-published*, 1994.

[10] Coffin, S.T., *Puzzle Craft*. Coffin, 1985.

[11] Slocum, J., & Sonneveld, D. *The 15 puzzle: how it drove the world crazy*. Slocum Puzzle Foundation, 1880.

[12] Coffin S., *Geometric puzzle design*. CRC Press, 2006.

[13] Stiny G., *Shape: talking about seeing and doing*, The MIT Press, 2008.

[14] Ozgan, S. Y., Bacinoglu, S. Z., & Coskun, E. The Potentials of Rule Based Approaches in De-sign Formation Processes. *Critical Spaces: Contemporary Perspectives in Urban, Spatial and Landscape Studies*, 2014, 13, 51.

[15] Wang, P. S. P. Three-dimensional sequential/parallel universal array grammars for polyhedral object pattern analysis. *International journal of pattern recognition and artificial intelligence*, 1994, *8*(02), 563-576.

[16] Alber, R., & Rudolph, S. On a grammar-based design language that supports automated design generation and creativity. In *Knowledge Intensive Design Technology* (pp. 19-35). Springer US, 2004.

[17] Thaller, W., Krispel, U., Zmugg, R., Havemann, S., & Fellner, D. W. Shape grammars on convex polyhedra. *Computers & Graphics*, 2013, *37*(6), 707-717.

[18] Buekenhout, F., & Parker, M., The number of nets of the regular convex polytopes in dimension $\leqslant$ 4. *Discrete mathematics*, 1998, 186(1), 69-94.

[19] Weisstein, Eric W. "Net." *MathWorld --A Wolfram Web Resource.* http://mathworld.wolfram.com/Net.html